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COMMENT

Hull percolation and standard percolation

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Abstract. We show that the 'hull percolation' problem considered recently by Roux *et al* is in one-to-one correspondence with standard bond percolation. Their random tiling of the plane by squares with circles is identical with the Baxter-Kelland-Wu polygon decomposition associated with clusters of the Q-state Potts model, which in the $Q \to 1$ limit gives percolation.

Roux *et al* (1988) considered recently the problem of random tiling of the plane by two kinds of squares having two quarters of circles drawn on them (figure 1). The lines formed by the concatenation of the fraction of circles in the plane are wavy loops which are self- and mutually-avoiding and actually span a square lattice. In the continuum limit they become fractal and the authors find numerically that this tiling has the same universal geometrical properties as various models of statistical mechanics, namely 'smart kinetic walks' (SKW) (Weinrib and Trugman 1985), hulls of percolation clusters (Leath and Reich 1978, Ziff 1986, Saleur and Duplantier 1987) and finally polymers at the Θ point (Coniglio *et al* 1987, Duplantier and Saleur 1987). They also study transport properties of the connected lines of the tiling. They call this random tiling problem a 'hull percolation', using the indirect relation of SKW to hull perimeters in percolation (Weinrib and Trugman 1985). Searching for a more direct analogy to percolation, they introduce a 'non-standard' percolation problem by considering the bonds generated by the diagonals of the plaquettes, which are left free by the quarters of circles (figure 1). They put onto each plaquette of the square lattice one of these diagonal or antidiagonal bonds each with probability $p_c = \frac{1}{2}$. They remark that this does not seem to be a percolation problem, strictly speaking, since one imposes a fixed occupancy level instead of leaving it free. They nevertheless observe that these bonds (diagonal or antidiagonal) form clusters of bonds on two lattices $\mathscr{L}_A$ and $\mathscr{L}_B$ dual of each other (see their figure 7). On both dual lattices $\mathscr{L}_A$ and $\mathscr{L}_B$ they remark that they

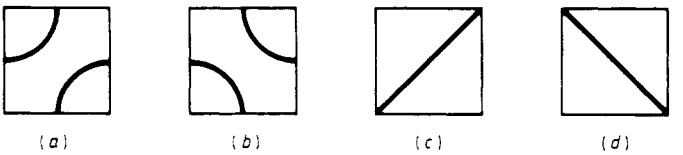

Figure 1. The elementary tiles of the plane (a, b) and their equivalent bonds (c, d).

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have two dual percolation configurations at $p_c(=\frac{1}{2})$. Their hulls are in direct correspondence to the loops drawn by the tiling model.

The purpose of this comment is to state that this geometrical tiling model is in exact correspondence with standard bond percolation on a single square lattice ($\mathscr{L}_A$ above for instance). More precisely, the random tiling by quarters of circles (figure 1) is just the usual Baxter *et al* (1976) polygon decomposition of the surrounding lattice, generated in one-to-one correspondence by percolating clusters (on $\mathscr{L}_A$), or more generally in the Potts model (Wu 1982). The two dual sets of clusters on lattices $\mathscr{L}_A$ and $\mathscr{L}_B$ of Roux *et al* are redundant. As we shall see, one of them (e.g. on $\mathscr{L}_A$) is sufficient to generate all the hulls, i.e. all the tiling, the other clusters on $\mathscr{L}_B$ being obtained immediately as a dual configuration, without any supplementary statistical freedom. So once the clusters are decided on $\mathscr{L}_A$, those of $\mathscr{L}_B$ are automatically frozen by duality. This property in the standard Q-state Potts model on a square lattice $\mathscr{L}_A$, which in the $Q \to 1$ limit reproduces percolation (Kasteleyn and Fortuin 1969), is actually used to find the critical point by self-duality (Wu 1982).

Let us now recall briefly the Baxter *et al* polygon decomposition of the standard Potts model, i.e. of the standard percolation, and show how the construction of the tiling model appears there. The Potts Hamiltonian is $\beta H = -\beta \sum_{\langle i,j \rangle} \delta_{\sigma_i \sigma_j}$ where the Potts variables take the values $\sigma_i = 1, \ldots, Q$ (integer), $\langle i, j \rangle$ being next-neighbour sites of a square lattice $\mathscr{L}_A$. It is well known that one can write the high-temperature expansion of the partition function as

$$Z_{\text{Potts}} = \sum_{\{\sigma\}} e^{-\beta H} = \sum_{\mathscr{G}} W(\mathscr{G}) = \sum_{\mathscr{G}} (e^{\beta} - 1)^B Q^C \tag{1}$$

where $W(\mathscr{G})$ is the weight of a graph $\mathscr{G}$ drawn on $\mathscr{L}_A$ made of clusters of occupied bonds, with a total number B of bonds, and a total number C of connected components (clusters) including all the isolated sites of $\mathscr{L}_A$ which do not belong to occupied bonds (figure 2). Therefore all sites of $\mathscr{L}_A$ either belong to occupied bonds or form isolated point clusters so the lattice $\mathscr{L}_A$ is totally spanned by $\mathscr{G}$. This remark will play a role later. The expression (1) now defines a model for real Q, and possesses a critical point for $Q \in [0, 4]$. Note that defining

$$p = 1 - e^{-\beta} \tag{2}$$

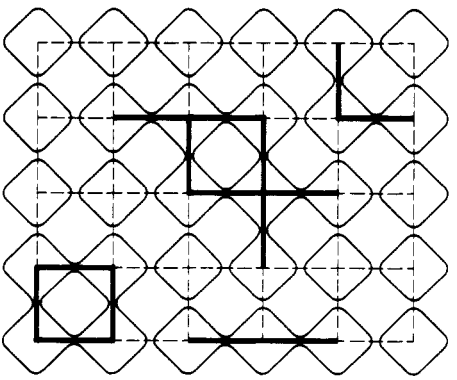

Figure 2. A cluster configuration $\mathscr{G}$ made of occupied bonds and of isolated sites on the square lattice $\mathscr{L}_A$ (broken lines), and the associated polygon decomposition of the diagonal surrounding lattice $\mathscr{S}$ of $\mathscr{L}_A$. $\mathscr{S}$ is obtained by joining nearest-neighbouring midpoints of $\mathscr{L}_A$.

we have identically

$$Z_{\text{Potts}} = p^{-\mathcal{N}} \sum_{\mathcal{G}} p^{\mathcal{N}-B}(1-p)^B Q^C \tag{3}$$

where $\mathcal{N}$ is the total number of bonds of $\mathcal{L}_A$. Hence for $Q=1$ we recover the bond percolation partition function on the square lattice. Now, a spanning graph $\mathcal{G}$ on $\mathcal{L}_A$ can be associated with a polygon decomposition of the surrounding lattice $\mathcal{S}$ of $\mathcal{L}_A$ (figure 2). The surrounding lattice $\mathcal{S}$ is the diagonal square lattice obtained by joining the midpoints of the edges of $\mathcal{L}_A$, when they are nearest neighbours. Note that the edges of $\mathcal{L}_A$ and of its dual $\mathcal{L}_B$ cross precisely at these midpoints, which are thus the common points of $\mathcal{L}_A$ and $\mathcal{L}_B$. Hence the surrounding lattice of $\mathcal{L}_B$ is also $\mathcal{S}$. Now, in the polygon decomposition of the surrounding lattice $\mathcal{S}$, associated with a spanning graph $\mathcal{G}$, the rule is that some vertices of $\mathcal{S}$ are cut open to let the bonds of $\mathcal{G}$ ($\in \mathcal{L}_A$) go through unintersected. This also applies along all the edges of the dual $\mathcal{L}_B$ which do not cross the bonds of $\mathcal{G}$ (figure 2). Note that the resulting polygon configuration appears to be made of islands, supporting the connected pieces of $\mathcal{G}$, surrounded by lakes where all the sites of the dual lattice $\mathcal{L}_B$ are immersed. Note that the isolated sites of $\mathcal{G}$ are also surrounded by small polygons. Now, on the lattice $\mathcal{L}_A$, having a total number of sites S, one has the Euler relation for any spanning graph $\mathcal{G}$

$$L = B + C - S \tag{4}$$

where L is the number of loops inside the clusters. The total number of polygons of the polygon decomposition of $\mathcal{S}$ is then

$$P = L + C \tag{5}$$

since besides L interior polygons, there is one external polygon for each of the C connected components of $\mathcal{G}$. From (4) and (5) one finds $C = (P - B + S)/2$ and one rewrites (1) as

$$Z_{\text{Potts}} = Q^{S/2} \sum_{\mathcal{G}} [(e^{\beta} - 1)Q^{-1/2}]^B Q^{P/2}. \tag{6}$$

The critical point of the Potts model is then known by self-duality to be (see, e.g., Wu 1982)

$$(e^{\beta_c} - 1)Q^{-1/2} = 1 \tag{7}$$

and corresponds to a weight independent of the number of occupied bonds, i.e. to a simple polygon model

$$Z_{\text{Potts critical}} = Q^{S/2} \sum_{\mathcal{G}} Q^{P/2}. \tag{8}$$

Note that the condition (7) corresponds on the dual lattice $\mathcal{L}_B$ to a Potts model which is also critical. Occupied bonds on $\mathcal{L}_A$ prevent the dual orthogonal edges of $\mathcal{L}_B$ from being occupied (figure 2). The critical point is self-dual and occurs when the bonds on $\mathcal{L}_A$ and $\mathcal{L}_B$ have the same probability, $p = p_c = \frac{1}{2}$. Note finally that for percolation $Q=1$, hence all polygon decompositions at the critical point have the same weight 1 in (8). Let us finally make contact with the random tiling by Roux *et al* (figure 3). The lattice they are tiling with the decorated squares of figure 1 is the diagonal lattice $\mathcal{S}^*$, dual of the surrounding lattice $\mathcal{S}$ of $\mathcal{L}_A$ and $\mathcal{L}_B$. The diagonal bonds of the tiles of figure 1 are bonds of $\mathcal{L}_A$ or $\mathcal{L}_B$. But the polygon decompositions of $\mathcal{S}$ associated with all spanning graphs $\mathcal{G}$ of $\mathcal{L}_A$ at β_c for $Q=1$ then generate all the tilings of the plane with equal probability, one lattice $\mathcal{L}_A$ being sufficient. The surrounding lattice

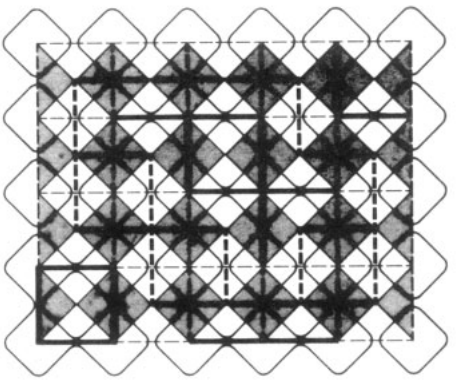

Figure 3. The passage to the random tiling. The tiled lattice is the diagonal square lattice $\mathscr{S}^*$, dual of the surrounding lattice $\mathscr{S}$. The bold broken line represents the dual graph $\mathscr{G}^*$ of $\mathscr{G}$ on the dual lattice $\mathscr{L}_B$. Both the bonds of $\mathscr{G}$ and $\mathscr{G}^*$ are bonds of figure 1. When $\beta \neq \beta_c$, the black and white tiles of $\mathscr{S}^*$ become anisotropic, the black preferring for $\beta > \beta_c$ the vertical tiling, the white the horizontal.

of $\mathscr{L}_A$ and $\mathscr{L}_B$ being the same $\mathscr{S}$, the dual clusters on $\mathscr{L}_B$ of $\mathscr{G}$ generate the same polygon decomposition, hence the same tiling (figure 3).

As a consequence the quoted authors were wondering how to escape from criticality in their random tiling model. Let us consider the effect of the standard deviation from criticality generated by the inverse temperature β in the Potts model. According to (6) and (7) when $\beta \neq \beta_c$, the number of occupied bonds on the lattice $\mathscr{L}_A$ becomes relevant. For instance, for $\beta > \beta_c$ (low-temperature phase) $(e^{\beta} - 1)Q^{-1/2} > 1$, and configurations $\mathscr{G}$ with more occupied bonds are favoured. This also means on the dual lattice $\mathscr{L}_B$, the dual configuration of $\mathscr{G}$ has less occupied bonds. To understand the effect on the diagonal tiled lattice $\mathscr{S}^*$, one has to decompose it into a checkerboard lattice, where (say) the black diagonal squares have a vertical diagonal belonging to $\mathscr{L}_A$, and the white a vertical belonging to $\mathscr{L}_B$. Then for $\beta > \beta_c$, configurations with more occupied bonds on $\mathscr{L}_A$ correspond to black squares of $\mathscr{S}^*$ tiled more favourably by vertical quarters of circles, and white squares more favourably by horizontal, the situation being reversed for $\beta < \beta_c$.

In conclusion, let us stress that one percolating lattice is sufficient for tiling the plane, because isolated sites, not only occupied bonds, are considered and surrounded by polygons. These isolated sites are really there in the geometrical properties of the Potts model, but one does not think about them in the usual formulation of percolation. This is perhaps the reason why the authors also introduced the dual lattice. Note also that the Baxter *et al* polygon decomposition is just the method we used (Saleur and Duplantier 1987) for calculating the exact fractal dimension $\frac{7}{4}$ of the hull.

We conclude that their study, rather than presenting a new model of percolation by tiling, appears to be an original numerical study of several critical properties of hulls of usual percolation clusters. In particular, by making all percolation clusters insulators, they have access to the conductivity of the hull. Note also that the question of the equivalence of hulls to polymers at the standard Θ point is still in debate (Poole *et al* 1988, Duplantier and Saleur 1988), while they are certainly equivalent to polymers in the presence of annealed obstacles.

Note finally that some other references are related to this discussion. Gunn and Ortũno (1985) considered random walks in a random environment, which for certain

weights are bouncing on percolation clusters. In this case, their trajectories are then the polygon lines of the hull. Grassberger (1986) similarly simulated the hull of bond percolation on the square lattice by random walks turning left or right with equal probabilities, and was able to get a good estimate of $D_H = 1.750 \pm 0.002$, in excellent agreement with the present exact $\frac{7}{4}$. He further considered the chance to 'survive' n steps before closing a loop n^{-c}. He gives $c = (\beta/\nu)D_H \approx 0.06$, while the exact value should be $c = (d - D_H)D_H^{-1} = \frac{1}{7} \simeq 0.14$ (Roux 1988), i.e. as in polymer physics $c = \nu d - 1$, for the probability of closing a loop, with here $\nu = \nu_\Theta = \frac{4}{7}$. Grassberger (unpublished) observed $c = 0.13$ numerically.

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